MATH 134A Review: System of Linear Equations

Facts to Know

A system of linear equations with two equations and two variables is

\[
\begin{align*}
ax + by &= u \\
cx + dy &= v
\end{align*}
\]

Here \(x, y\) are variables, and \(a, b, c, d, u, v\) are constants.

The corresponding matrix equation is

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
u \\
v
\end{bmatrix}
\]

A solution is a vector, and in particular, for this setting, a solution is of the form \(\begin{bmatrix}x_0 \\ y_0\end{bmatrix}\) and \(ax_0 + by_0 = u\) and \(cx_0 + dy_0 = v\).

A method to find a solution is:

1. Find \(\begin{bmatrix}a & b \\ c & d\end{bmatrix}^{-1}\)
2. Multiply \(\begin{bmatrix}a & b \\ c & d\end{bmatrix}^{-1}\) on the left of both sides \(A \vec{x} = \vec{b}\).

Recall the inverse of a \(2 \times 2\) matrix is

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc}
\begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]
Examples

1. A restaurant manager wants to purchase 200 sets of dishes. One design costs $25 per set, while another costs $45 per set. If she only has $7400 to spend, how many of each design should be ordered?

\[
\begin{align*}
\begin{bmatrix}
X \\
Y
\end{bmatrix} & = \begin{bmatrix}
\frac{1}{45 - 25} \\
\frac{1}{-25}
\end{bmatrix}
\begin{bmatrix}
45 & -1 \\
-25 & 1
\end{bmatrix}
\begin{bmatrix}
200 \\
7400
\end{bmatrix} \\
\begin{bmatrix}
X \\
Y
\end{bmatrix} & = \frac{1}{20}
\begin{bmatrix}
45 & -1 \\
-25 & 1
\end{bmatrix}
\begin{bmatrix}
9000 \\
7400
\end{bmatrix} \\
& = \frac{1}{20}
\begin{bmatrix}
1600 \\
2400
\end{bmatrix} = \begin{bmatrix} 80 \\ 120 \end{bmatrix}
\end{align*}
\]

80 dishes bought which costs $25
120 dishes bought which costs $45
2. A movie theater charges $9.00 for adults and $7.00 for senior citizens. On a day when 325 people paid an admission, the total receipts were $2495. How many who paid were adults? How many were seniors?

\[ \begin{align*}
\begin{cases}
  x + y &= 325 \\
  9x + 7y &= 2495
\end{cases}
\end{align*} \]

\[ \begin{align*}
\begin{bmatrix}
  9 & 7 \\
  7 & 9
\end{bmatrix}^{-1} &= \frac{1}{7-9} \begin{bmatrix}
  7 & -1 \\
  -9 & 1
\end{bmatrix} \\
&= \frac{1}{-2} \begin{bmatrix}
  7 & -1 \\
  -9 & 1
\end{bmatrix} = A^{-1}
\end{align*} \]

\[ A \begin{bmatrix}
  x \\
  y
\end{bmatrix} = b \]

\[ \begin{bmatrix}
  x \\
  y
\end{bmatrix} = \frac{1}{-2} \begin{bmatrix}
  7 & -1 \\
  -9 & 1
\end{bmatrix} \begin{bmatrix}
  325 \\
  2495
\end{bmatrix} \]

\[ = \frac{1}{-2} \begin{bmatrix}
  (325 - 2495) \\
  (-2925 + 2495)
\end{bmatrix} \]

\[ = \frac{1}{-2} \begin{bmatrix}
  -220 \\
  -430
\end{bmatrix} \]

\[ = \begin{bmatrix}
  110 \\
  215
\end{bmatrix} \]

110 adults and 215 senior citizens